Tensor Products

Definition of Tensor: Tensor refers to objects with multiple indexes. Comparatively, a “vector” has one index while a “scalar” has none.Assume $V$and W are in a vector space, a pure tensor is an element of $V \otimes W$ of the form $v \otimes w$ where $v \in V$ and $w \in W$

What is a Tensor Product?

With a tensor product, we can construct a big vector space out of at least two smaller vector spaces.

If we start with two vector spaces V, n-dimensional, and W, m-dimensional, the tensor products of these two spaces would be nm-dimensional.

Explanation:

V is $\left\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\right\}$ and W is $\left\{\vec{f}_1, \vec{f}_2, ..., \vec{f}_m\right\}$

We define nm basis vectors $\vec{e}_i \otimes \vec{f}_j$, where i = 1,⋯,n and j= 1,⋯ ,m

For two vector spaces, the tensor product is bilinear, meaning it is linear in V and linear in W as well.

$$\vec{v} \otimes \vec{w} = \left(\sum_i^n v_i \vec{e}_i\right) \otimes \left(\sum_j^m w_j \vec{f}_j\right) = \sum_i^n \sum_j^m v_i w_j \left(\vec{e}_i \otimes \vec{f}_j\right)$$

Example: Let n=2 and m=3 the tensor product would be nm-dimensional, so for this example, 6 dimensional. The basis vectors are $\vec{e}_1 \otimes \vec{f}_1, \vec{e}_1 \otimes \vec{f}_2, \vec{e}_1 \otimes \vec{f}_3, \vec{e}_2 \otimes \vec{f}_1,$

$\vec{e}_2 \otimes \vec{f}_2, \vec{e}_2 \otimes \vec{f}_3$

We can write these as six-component column vectors as below.

$$\vec{e}_1 \otimes \vec{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_1 \otimes \vec{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_1 \otimes \vec{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 \otimes \vec{f}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 \otimes \vec{f}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 \otimes \vec{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{pmatrix}.$$

The tensor product is shown below for general vectors $\vec{v}$ and $\vec{w}$

$$\vec{v} \otimes \vec{w} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{pmatrix}.$$

Matrices: (Kronecker Products):

As we are interested in application in mechanics, we are focused on

$$\vec{v} \longmapsto A\vec{v},$$

and

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V.$$

We can write the matrix as $A \otimes I$, I being the identity matrix.We will return to our prior example where n=2 and m=3. The A matrix would be a two-by-two and $A \otimes I$ is a six-by-six.

$$A \otimes I = \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array}\right).$$

We will act this expression on $\vec{v} \otimes \vec{w}$, and yield:

$$(A \otimes I)(\vec{v} \otimes \vec{w}) = \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array}\right) \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{pmatrix} = \begin{pmatrix} (a_{11} v_1 + a_{12} v_2) w_1 \\ (a_{11} v_1 + a_{12} v_2) w_2 \\ (a_{11} v_1 + a_{12} v_2) w_3 \\ \hline (a_{21} v_1 + a_{22} v_2) w_1 \\ (a_{21} v_1 + a_{22} v_2) w_2 \\ (a_{21} v_1 + a_{22} v_2) w_3 \end{pmatrix} = (A\vec{v}) \otimes \vec{w}.$$

In this scenario, the matrix A only acts on $\vec{v}$ but does not impact $\vec{w}$.

This is also evident in matrix B: $W \to W$ maps $\vec{w} \to B\vec{w}$. Matrix $B \otimes I$ will only act on $\vec{w}$,

$$I \otimes B = \left(\begin{array}{ccc|ccc} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{array}\right).$$

$$(I \otimes B)(\vec{v} \otimes \vec{w}) = \left(\begin{array}{ccc|ccc} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{array}\right) \left(\begin{array}{c} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{array}\right) = \left(\begin{array}{c} v_1(b_{11}w_1 + b_{12}w_2 + b_{13}w_3) \\ v_1(b_{21}w_1 + b_{22}w_2 + b_{23}w_3) \\ v_1(b_{31}w_1 + b_{32}w_2 + b_{33}w_3) \\ \hline v_2(b_{11}w_1 + b_{12}w_2 + b_{13}w_3) \\ v_2(b_{21}w_1 + b_{22}w_2 + b_{23}w_3) \\ v_2(b_{31}w_1 + b_{32}w_2 + b_{33}w_3) \end{array}\right) = \vec{v} \otimes (B\vec{w}).$$

From above calculations, we can conclude a general rule: $(A \otimes I)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes \vec{w}$ and $(B \otimes I)(\vec{v} \otimes \vec{w}) = \vec{v} \otimes (B\vec{w})$

Helpful Proposition: $(A \otimes B)(C \otimes D) = (AC \otimes BD)$

In the case where we have two matrices, their multiplications are done on each vector space separately, as shown below:

$(A_1 \otimes I)(A_2 \otimes I) = (A_1 A_2) \otimes I,$
$(I \otimes B_1)(I \otimes B_2) = I \otimes (B_1 B_2)$
$(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) = (A \otimes B)$

The expression for $(A \otimes B)$ allows us to write it in the matrix form explicitly:

$$A \otimes B = \left(\begin{array}{ccc|ccc} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} \end{array}\right).$$

We can verify that $(A \otimes B)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes (B\vec{w})$

Properties of Tensor (Kronecker) Products:

Pulling Across Scalars

$\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$

Example: $\mathbb{R} \otimes V$

$2(3 \otimes u) + 5(1 \otimes v)$

$= (3 * 1) \otimes (2u) + 1 \otimes (5v)$

$= 1 \otimes (6u) + 1 \otimes (5v)$

$= 1 \otimes (6u + 5v)$

So, we can always end up combining the tensor products by pulling across scalars

Theorem: If $\vec{v}$ is an eigenvector for A with eigenvalue of λ, and $\vec{w}$ is an eigenvector for B with eigenvalue of σ, then $(\vec{v} \otimes \vec{w})$ is an eigenvector for $(A \otimes B)$.

Proof: $(A \otimes B)(\vec{v} \otimes \vec{w}) = A\vec{v} \otimes B\vec{w} = \lambda\vec{v} \otimes \sigma\vec{w} = \lambda\sigma(\vec{v} \otimes \vec{w})$

Corollary 1: Tr $(A \otimes B) = (TrA)(TrB)$

Proof: Tr $(A \otimes B) = \Sigma\, eigenvalues\, of (A \otimes B) = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_i \sigma_j = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} \lambda_i \sigma_j = (TrA)(TrB)$

Corollary 2: $det(A \otimes B) = (detA)^n (detB)^m$

Proof: $det(A \otimes B) = \prod$ eigenvalues l of $(A \otimes B) = \prod_{i=1}^{m}\prod_{j=1}^{n} \lambda_i \sigma_j = \prod_{i=1}^{m} (\lambda_i^{\,n} \prod_{j=1}^{n} \sigma_j) = \prod_{i=1}^{m} \lambda_i^{\,n} \prod_{j=1}^{n} \sigma_j^{\,m}$

$= (detA)^n (detB)^m$